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The Equations of Strata for Binary Forms

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Let g be a fixed integer. We consider the space X_g of binary forms of degree g. We write $X_g = \operatorname{Spec} R_g$ where $R_g = S.(S_g V)$, V being a two-dimensional vector space. Let $X_{p,g} \subset X_g$ be the subset of binary forms having a root of multiplicity $\geqslant p$. We consider the ideal J_p of polynomial functions vanishing on $X_{p,g}$. For $p \geqslant [g/2]+1$, $X_{p,g}$ is a stratum in the sense of Hesselink [H]. For $p = [g/2]+1$, $X_{p,g}$ is a null-cone, so J_p becomes the radical of the ideal generated by $SL(V)$-invariants of positive degree in R_g. We describe explicitly the generators of J_p for $p > [g/2]+1$. The surprising result is that those generators occur in degrees $\leqslant 4$. We also describe explicitly the Hilbert functions of R_g/J_p and the decomposition of R_g/J_p into representations of $SL(V)$. We denote by $S_{(a,b)}V$ the space $S_{a-b}V \otimes (\Lambda^2 V)^b$.

Let $R_g = S.(S_g V)$, V-vector space over $\mathbb{C}$, $\dim V = 2$, and let α, β be the basis of V. Let J_p be the ideal of elements of R_g vanishing on $X_{p,g} (p \geqslant [g/2]+1)$. By the result of Hesselink [H] the space $\tilde{X}_{p,g} = \operatorname{Proj}(R_g/J_p)$ has the following desingularization $Y_{p,g}$

$$\begin{array}{l} Y_{p,g} = \{(R, f) \in \mathbb{P}(V) \times X_g \mid f \text{ has a root of multiplicity } p \text{ at } R\} \\ \Big\downarrow \pi \\ \tilde{X}_{p,g} \end{array}$$

We can think of $\mathbb{P}(V)$ as the grassmannian with the tautological sequence $0 \to R \to V \to Q \to 0$ ($\dim R = \dim Q = 1$). Then $Y_{p,g} \subset \mathbb{P}(V) \times X_g$ and we can treat $\mathcal{O}_{Y_{p,g}} = S.(T_p)$ where $T_p = Q^p \otimes S_{g-p} V$ is a factor of $S_g V$. Then we know again from [H] that the normalization

$$\overline{R_g/J_p} = \pi_* S.(T_p).$$

One should observe that geometrically the normalization $\overline{X}_{p,g} =$

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$\{(l, f) \mid l \in \mathbb{P}(V), f \in X_{g-p}\}$ and the map $\bar{X}_{g,p} \to X_{g,p}$ is given by $(l, f) \to l^p f$. Using Serre's theorem we find out that

$$\overline{R_g/J_p} = \sum_{d \geqslant 0} S_{dp}(V) \otimes S_d(S_{g-p} V).$$

Now let us consider the composition $R_g \to R_g/J_p \to \overline{R_g/J_p}$. It is clearly the morphism θ

$$S_d(S_g V) \xrightarrow{\theta} S_{dp} V \otimes S_d(S_{g-p} V)$$

$$\theta(u_1 \cdots u_d) = \sum_{i_1 - i_d} \delta^1_{i_1}(u_1) \cdots \delta^1_{i_d}(u_d) \otimes \delta^2_{i_1}(u_1) \cdots \delta^2_{i_d}(u_d),$$

where $\delta(u_j) = \sum \delta^1_i(u_j) \otimes \delta^2_i(u_j)$ is the diagonalization $S_g V \to S_p V \otimes S_{g-p} V$.

Let $C_{g,p}(V) = (\operatorname{Coker} \theta)$ and let $C_{g,p,d}(V)$ be its dth component. To calculate the Hilbert function of R_g/J_p it is enough to calculate the dimensions of $C_{g,p,d}(V)$.

THEOREM 1. $C_{g,p,d}(V) = S_{d(p+1)-1,1} V \otimes S_d(S_{g-p-1} V)$.

Let us consider the sequence

$$S_d(S_g V) \xrightarrow{\theta} S_{dp} V \otimes S_d(S_{g-p} V)$$

$$\xrightarrow{\psi} S_{d(p+1)-1,1} V \otimes S_d(S_{g-p-1} V) \longrightarrow 0, \qquad (*)$$

where $\psi(x \otimes u_1 \cdots u_d) = \sum pr[x \otimes \delta^1_{i_1}(u_1) \cdots \delta^1_{i_d}(u_d)] \otimes \delta^2_{i_1}(u_1) \cdots \delta^2_{i_d}(u_d)$ $pr(x \otimes y)$ is the projection of $S_{dp} V \otimes S_d V$ onto $S_{d(p+1)-1,1} V$. First of all

$$pr(x \otimes y) = \begin{vmatrix} \dfrac{\partial x}{\partial \alpha} & \dfrac{\partial x}{\partial \beta} \\ \dfrac{\partial y}{\partial \alpha} & \dfrac{\partial y}{\partial \beta} \end{vmatrix}.$$

$S_d(S_g V)$ is generated linearly by the elements $(l_1^g) \cdots (l_d^g)$ where $l_1, \ldots, l_d$ are linear forms. Then we calculate easily that $\psi\theta(l_1^g) \cdots (l_d^g) = 0$, so $\psi\theta = 0$. Thus to prove Theorem 1 it is enough to show that $(*)$ does not have homology in the middle and that ψ is a surjection. In order to obtain this information we have to consider the syzygies of $\overline{R_g/Y_p}$ and $C_{g,p}$ as R_g-modules. Let S_p be the kernel

$$0 \longrightarrow S_p \longrightarrow S_g V \longrightarrow T_p \longrightarrow 0.$$

Then we see readily that $S_p = R^{g-p+1} \otimes S_{p-1} V$. Thus, by the techniques of

[J–P, J–P–W] the syzygies of R_g/J_p are given by higher direct images of $\Lambda.(S_p)$. Thus the chain of syzygies of R_g/J_p looks like this

$$0 \longleftarrow R_g/J_p \longleftarrow S_0 V \oplus S_{(g-p,1)} V \otimes S_{p-1} V \longleftarrow S_{(2g-2p+1,1)} V \otimes \Lambda^2(S_{p-1} V)$$

$$\longleftarrow \cdots \longleftarrow S_{(jg-jp+j-1,1)} V \otimes \Lambda^j(S_{p-1} V) \longleftarrow \cdots .$$

Now we describe the syzygies of the module $M_{g,p} = \sum_{d \geqslant 0} S_{d(p+1)-1,1} V \otimes S_d(S_{g-p-1} V)$. We consider the module $M_{g,p} = \pi_*(Q^* \otimes R \otimes m)$ where m is the maximal ideal in $S.(T_{p+1})$. We see using Serre's theorem that $M_{g,p,d} = S_{d(p+1)-1,1} V \otimes S_d(S_{g-p-1} V)$. The chain of syzygies of $M_{g,p}$ is given by $\Lambda.(S_{p+1}) \otimes Q^{-1} \otimes R \otimes \Lambda^+(T_{p+1})$. In fact, the jth bundle of the syzygies of $Q^* \otimes R \otimes m$ on $Y_{g,p}$ can be represented as follows:

$$0 \longrightarrow Q^{-1} \otimes R^{(g-p)j+1} \otimes \Lambda^j(S_p V)$$

$$\longrightarrow Q^{-1} \otimes R \otimes \Lambda^j(S_g V) \longrightarrow \operatorname{syz} j \longrightarrow 0.$$

Thus $H^0(\operatorname{syz} j) = \ker(S_{j(g-p)} V \otimes \Lambda^j(S_p V) \longrightarrow \Lambda^j(S_g V))$

$$H^1(\operatorname{syz} j) = \operatorname{Coker}(S_{j(p-p)} V \otimes \Lambda^j(S_p V) \longrightarrow \Lambda^j(S_g V)).$$

The syzygies of $M_{g,p}$ are given by:

$$\mathrm{SYZ}\, j = H^0(\operatorname{syz} j) \oplus H^1(\operatorname{syz}(j-1)).$$

Remark. The resolution given by SYZ j need not be minimal, but this is no important for our goals. To get the presentation of $M_{g,p}$ we need the following:

LEMMA 2. $\operatorname{Coker}(S_{3(g-p)} V \otimes \Lambda^3(S_p V) \longrightarrow \Lambda^3(S_g V)) = 0$ *for* $p \geqslant [g/2] + 1$.

Proof. First of all, let $E_i = \alpha^i \beta^{g-i}$ $(i = 0, ..., g)$ be the basis of $S_g V$. Then the elements $E_i \wedge E_j \wedge E_k$ $(0 \leqslant i < j < k \leqslant p)$ are in the image of our map. Indeed,

$$E_i \wedge E_j \wedge E_k = \operatorname{Im}(\beta^{3(g-p)} \otimes \alpha^i \beta^{p-i} \wedge \alpha^j \beta^{p-j} \wedge \alpha^k \beta^{p-k}).$$

Thus it is enough to show that those elements generate $\Lambda^3(S_g V)$ as $SL(V)$-module. Let N be $SL(V)$-submodule of $\Lambda^3(S_g V)$ generated by our elements. We prove the induction of l by the following statement

$$(St_l) \qquad E_i \wedge E_j \wedge E_k \qquad \text{for} \quad 0 \leqslant i < j < k \leqslant l \text{ belong to } N.$$

For $l = p$ (St_l) is true, for $l = g$ it is the statement of the lemma. We show that $(St_l) \Rightarrow (St_{l+1})$.

Let us assume (St_l) holds for some $l \geqslant p$. Let us consider $E_i \wedge E_j \wedge E_l$ and let us act on it by the element θ_x of $SL(V)$ sending α to α and β to $\beta + x\alpha$.

$$\theta_x(E_i \wedge E_j \wedge E_l) = \sum_{i=0}^{M} x^i U_i.$$

Then by the usual argument involving Vandermonde determinant we know that $U_i \in N$. But $U_1 = A(E_{i+1} \wedge E_j \wedge E_l) + B(E_i \wedge E_{j+1} \wedge E_l) + C(E_i \wedge E_j \wedge E_{l+1})$ for $A, B, C \neq 0$. Thus $E_i \wedge E_j \wedge E_{l+1} \in N$ for all $i < j < l$. It remains to deal with the elements $E_i \wedge E_l \wedge E_{l+1}$. We notice that $E_{l-1} \wedge E_l \wedge E_{l+1} \in N$. Indeed, acting by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we see it is enough to show that $E_{g-l-1} \wedge E_{g-l} \wedge E_{g-l+1} \in N$ which is true because $g - l + 1 \leqslant g - p + 1 \leqslant p$. Now we show by reverse induction on i that $E_i \wedge E_l \wedge E_{l+1} \in N$. Indeed, assuming $E_{i+1} \wedge E_l \wedge E_{l+1} \in N$, we act by η_x where $\eta_x(\alpha) = \alpha + x\beta$, $\eta_x(\beta) = \beta$. $\eta_x(E_{i+1} \wedge E_l \wedge E_{l+1}) = \sum_{i=0}^{M} x^i V_i$.

$V_1 = A(E_i \wedge E_l \wedge E_{l+1}) + B(E_i \wedge E_{l-1} \wedge E_{l+1})$, $A, B \neq 0$. Both V_1 and $E_i \wedge E_{l-1} \wedge E_{l+1}$ belong to N, so $E_i \wedge E_l \wedge E_{l+1}$ does. The lemma is thus proved.

Now we see that $M_{g,p}$ has the following presentation as $S.(S_g V)$-module:

$$\begin{aligned} 0 \longleftarrow M_{g,p} &\longleftarrow S_{p,1} \otimes S_{g-p-1} V \\ &\longleftarrow \operatorname{Ker}(S_{2(g-p)} V \otimes \Lambda^2(S_p V) \longrightarrow \Lambda^2(S_g V)). \end{aligned}$$

Now we come back to our sequence $(*)$. We want to show that $C_{g,p,d} = M_{g,p,d}$. But the right term is generated as R_g-module by $S_{p,1} \otimes S_{g-p-1} V$, and the middle term by $S_{p,1} V \otimes S_{g-p-1} V \oplus R$. Thus we get the sequence of R_g-modules

$$\begin{aligned} 0 \longrightarrow \operatorname{Ker} \psi &\longrightarrow \sum_{d \geqslant 0} S_{dp} V \otimes S_d(S_{g-p} V) \\ &\longrightarrow \sum_{d \geqslant 0} S_{d(p+1)-1,1} V \otimes S_d(S_{g-p-1} V) \longrightarrow 0. \end{aligned}$$

It remains to show that $\ker \psi$ is generated by R in degree 0. But the modules involved have the following presentations

$$\begin{tikzcd} 0 & R_g/J_p \arrow[l] \arrow[d, "\psi"] & S_0 V \oplus S_{(g-p,1)} V \otimes S_{p-1} V \arrow[l] \arrow[d, "\psi_1"] & S_{(2g-2p+1,1)} V \otimes \Lambda^2(S_{p-1} V) \arrow[l] \arrow[d, "\psi_2"] \\ 0 & M_{g,p} \arrow[l] & S_{(g-p,1)} V \otimes S_{p-1} V \arrow[l] & \operatorname{Ker}(S_{2(g-p)} V \otimes \Lambda^2(S_p V) \to \Lambda^2(S_g V)) \arrow[l] \end{tikzcd}$$

Now it is clear that ψ_1, ψ_2 are epimorphisms, so $\operatorname{Ker}\theta$ is generated by $S_0 V$ in degree 0. This shows Theorem 1 is true.

Now we are ready to prove the main result of this paper.

THEOREM 3. *The generators of the ideal J_p have degrees $\leqslant 4$ for $p \geqslant [g/2]+1$.*

Proof. It is enough to calculate the second module in the resolution of R_g/J_p. We use the basic diagram with exact columns.

$$\begin{tikzcd}
0 \arrow[r] & R_g/J_p \arrow[r] & \overline{R_g/J_p} \arrow[r] & C_{g,p} \arrow[r] & 0 \\
 & & S_0 V \oplus S_{(g-p,1)} V \otimes S_{p-1} V \arrow[u] \arrow[r] & S_{(g-p,1)} V \otimes S_{p-1} V \arrow[u] \arrow[r] & 0 \\
 & & S_{(2g-2p+1,1)} V \otimes \Lambda^2(S_{p-1} V) \arrow[u] \arrow[r] & \operatorname{Ker}(S_{2(g-p)} V \otimes \Lambda^2(S_p) \to \Lambda^2(S_g V)) \arrow[u] \arrow[r] & 0 \\
 & & S_{(3g-3p+2,1)} V \otimes \Lambda^3(S_{p-1} V) \arrow[u] \arrow[r] & \begin{matrix} \operatorname{Ker}(S_{3(g-p)} V \otimes \Lambda^3(S_p) \to \Lambda^3(S_g V)) \\ \oplus \\ \operatorname{Coker}(S_{4(g-p)} V \otimes \Lambda^4(S_p) \to \Lambda^4(S_g V)) \end{matrix} \arrow[u] & \\
 & & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

The resolution of R_g/J_p is the cone of the above map of complexes. We see that in degree 1 it has elements of degree $\leqslant 4$.

Remark. To describe the equations more explicitly let us note that, for example, for $X \in \Lambda^4(S_g V)$ the image of X in $S_0 V$ comes from the following diagram.

$$\begin{tikzcd}
S_0 V & & & & X_5 & & \\
S_{2g-2p+1,1} \otimes \Lambda^2(S_{p-1} V) \arrow[u] \arrow[r] & S_{2(g-p)} V \otimes \Lambda^2(S_p V) \arrow[r] & \Lambda^2(S_g V) \arrow[r] & 0 & X_4 \arrow[u] \arrow[r, squiggly] & X_3 & \\
 & S_{3(g-p)} V \otimes \Lambda^3(S_g V) \arrow[u] \arrow[r] & \Lambda^3(S_g V) \arrow[u] \arrow[r] & 0 & & X_2 \arrow[u] \arrow[r, squiggly] & X_1 \\
 & S_{4(g-p)} V \otimes \Lambda^4(S_g V) \arrow[u] \arrow[r] & \Lambda^4(S_g V) \arrow[u] & & & & X \arrow[u]
\end{tikzcd}$$

so $d_1(\bar{X}) = X_5$. Similarly for the elements of degree 3.

COROLLARY 4. (a) *The* dth *component of* R_g/J_p *in representation ring of* $SL(V)$ *equals* $S_{dp}V \otimes S_d(S_{g-p}V) - S_{d(p+1)-1,1}V \otimes S_d(S_{g-p-1}V)$.

(b) *The Hilbert function of* R_g/J_p *equals* $H(R_g/J_p, d) = \dim(R_g/J_p)_d = (dp+1)\binom{g-p+d}{g-p} - (d(p+1)-1)\binom{g-p+d-1}{g-p-1}$.

REFERENCES

[H] W. HESSELINK, Desingularizations of varieties of nullforms, *Invent. Math.* **55** (1979), 141–163.

[J-P-W] T. JÓZEFIAK, P. PRAGACZ, AND J. WEYMAN, Resolutions of determinantal varieties, *Astérisque* **87–88** (1981), 109–189.

[J-P] P. PRAGACZ AND J. WEYMAN, "On the Construction of Resolutions of Determinantal Ideals, A Survey," *in* Sem. Dubreil-Malliavin, LN 1220.